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Stability in Symmetrical Two-Dimensional Nonlinear and Time Varying Control Systems

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Abstract

An extended stability criterion for cross-coupled, symmetrical, two-dimensional, nonlinear and time-varying systems is presented. The effects of time variation as well as cross-coupling on the system's stability are discussed.

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I. Introduction

An extension of the results of Newman [1] to the nonlinear case was derived, in [5], using the frequency-domain approach of the Popov type. It was shown that the results of Lindgren and Pinkos [2] are unnecessarily restrictive by requiring the same slope for the Popov lines. It was also shown that stabilization of an otherwise unstable system can be achieved by the introduction of an appropriate cross coupling. The contribution of this paper is to extend the stability determination for cross-coupled symmetrical two-dimensional nonlinear systems illustrated in Fig. 1, to the case where the nonlinearities may be also time-varying.

The proof of the Theorem used here is based on a new lemma, which is developed for the first time, for symmetrical positive semi-definite matrix multipliers rather than diagonal ones. A technique is developed where stability expressions is achieved through mathematical manipulation on a transformed version of the problem rather than on the original.

II. System Description

Figure 1 shows the structure of the autonomous, continuous-time, symmetrical, two-dimensional, nonlinear and time-varying system under consideration. It is assumed that the following hold:

(A1) Each of the linear time-invariant, nonanticipative subsystems $S_i (i = 1,2,3,4)$, with $f_j[\sigma_j(t),t]$ $(j = 1,2)$ as an input and $y_i(t)$ as an output, is represented in the state space form as:

$$\hat{S}_i \left| \begin{array}{ll} \dot{\underline{x}}_i(t) = A_i \quad \underline{x}_i(t) - \underline{b}_i \; f_j[\sigma_j(t),t] & j=1 \text{ when: } i=1,2 \\ & \text{and} \\ & j=2 \text{ when: } i=3,4 \\ y_i(t) = \underline{c}_i^T \quad \underline{x}_i(t) \quad \forall \; i=1, \ldots ,4 & \end{array} \right.$$

where: A_i - n_i^{th} order Square, asymptotically stable matrices with real, time-invariant elements.

$\underline{b}_i, \underline{c}_i$ - n_i^{th} order Column vectors with real, time-invariant elements.

$\underline{x}_i$ - n_i^{th} order Vectors of the

$\sigma_j(t)$ - Scalar time functions.

In addition, it is further assumed that $[A_i, b_i]$'s are completely controllable and that $[A_i, c_i]$'s are completely observable.

(A2) The noninteracting nonlinear and time-varying elements are characterized by their individual input-output relations. The output of each nonlinearity is given by $f_j[\sigma_j(t),t]$ which is a real continuous function, such that:

$$\text{(a)} \qquad 0 < \sigma_j(t) f_j[\underline{\sigma}(t),t] < K_j \sigma_j^2(t), \; \forall\, \sigma_j(t) \neq 0,$$

$$\text{(b)} \qquad f_j(o,t) = 0$$

Preliminary Consideration

(1) Based on the linear subsystem description of (A1), the transfer function of each subsystem S_i is given by:

$$\left\{\hat{S}_i \;\middle|\; \hat{H}_i(s) = \underline{C}_i^T \quad (SI - A_i)^{-1} \quad \underline{b}_i\right\}, \; (i=1,2,3,4) .$$

However due to the symmetrical nature of the system: $S_1 = S_4$ and $S_2 = S_3$ so that the transfer function associated with the subsystem is as follows:

$$S_1 \rightarrow H_1, \; S_2 \rightarrow H_2, \; S_3 \rightarrow H_2, \; S_4 \rightarrow H_1 .$$

(2) The system in Fig. 1 is equivalent to the systems in Fig. 2,

where

$$\hat{W}(s) = \left[\begin{array}{c|c} \hat{H}_1(s) & \hat{H}_2(s) \\ \hline \hat{H}_1(s) & \hat{H}_1(s) \end{array}\right]$$

$$\underline{f}^T[\underline{\sigma}(t),t] = \{f_1[\sigma_1(t),t],\ f_2[\sigma_2(t),t]\} \ .$$

(3) To facilitate the analysis we apply the orthogonal transformation

$$T = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & -1 \end{array}\right] , \text{ (note: } T^T = T^{-1})$$

to the system's transfer matrix W to yield (see Lindgren[2] Rootenberg[5] a diagonal matrix $\hat{W}$. Simultaneously $\underline{\sigma}(t)$ and $\underline{f}[\underline{\sigma}(t),t]$ are transformed (see Fig. 3) to $\hat{\underline{\sigma}}(t)$ and $\hat{\underline{f}}[\hat{\underline{\sigma}}(t),t]$, respectively:

$$\underline{\sigma}(t) = T\ \hat{\underline{\sigma}}(t)$$

$$\underline{f}[\underline{\sigma}(t),t] = T \cdot \hat{\underline{f}}[\hat{\underline{\sigma}}(t),t] \ .$$

(4) In the sequel we will also define a system $\hat{S}$ as having the same structure as in Fig. 2 with W(s) replaced by $\hat{W}(s)$ and $\underline{f}[\underline{\sigma}(t),t]$ replaced by $\hat{\underline{f}}[\hat{\underline{\sigma}}(t),t]$.

III. Stability Criterion

The proof of the stability criterion depends heavily on the following two lemmas:

Lemma I: (Anderson & Moore[4])

Let $\hat{Z}(s)$ be an nxn matrix of real functions of a complex variable s, with $\hat{Z}(\infty) < \infty$. Let $\{F,G,M,J\}$ be a realization for $\hat{Z}(s)$, in the sense that:

$$\hat{Z}(s) = J + M(sI-F)^{-1}\hat{G} ,$$

with F square and of minimal dimension

If $\hat{Z}(s)$ is positive real, $(J+J^T)$ nonsingular, $\hat{Z}(s)$ possess no $j\omega$-axis poles, and $\hat{Z}(j\omega)$ is positive definite.
Then, there exists a matrix $P(t) = \lim_{t_1\to\infty} \pi(t,t_1)$, where $\pi(t,t_1)$ is the solution of

$$-\dot{\pi}(t,t_1) = \pi[F-\hat{G}(J+J^T)^{-1}M^T] + [F^T-M(J+J^T)^{-1}\hat{G}^T]\,\pi + \pi\hat{G}(J+J^T)^{-1}G^T\pi + M(J+J^T)^{-1}M$$

with $\pi(t_1,t_1) = 0$.

Lemma II: The system of Fig. 6 is stable if there exist symmetrical, positive semi-definite matrices $\hat{K} \geq 0$, $\hat{\alpha} \geq 0$, $\hat{\beta} \geq 0$, transformable (by a linear time-invariant and non-singular transformation T such that $T^t = T^{-1}$) to a diagonal form:

$$\alpha \hat{=} T \cdot \hat{\alpha} \cdot T^{-1} = \mathrm{diag}\{\alpha_1 \ldots \alpha_n\}$$

$$\beta \hat{=} T \cdot \hat{\beta} \cdot T^{-1} = \mathrm{diag}\{\beta_1 \ldots \beta_n\}$$

$$K \hat{=} T \cdot \hat{K} \cdot T^{-1} = \mathrm{diag}\left\{\frac{1}{K_1} \ldots \frac{1}{K_n}\right\}$$

Such that: $\alpha_i \geq 0, \beta_i \geq 0$, $\alpha_i + \beta_i > 0$ and $-\frac{\alpha_i}{\beta_i}$ is not a pole of any element of the i^{th} row of W(s), where

$$W(s) \triangleq T \cdot \hat{W}(s) \cdot T^{-1} = \mathrm{diag}\{H_1(s), \ldots, H_n(s)\}$$

and,

$$\hat{Z}(s) \triangleq \hat{\alpha} \cdot \hat{K} + (2+\beta \cdot s)W(s) \quad \text{is positive real.} \qquad (1)$$

Provided that:

$$\hat{\underline{f}}^T(\hat{\underline{\sigma}}_1 t) \cdot \hat{\alpha} \cdot \left\{\hat{\underline{\sigma}} - \hat{K} \cdot \hat{\underline{f}}(\hat{\underline{\sigma}}_1 t)\right\} > \int_{\underline{0}}^{\underline{\sigma}} \left[\frac{\partial \hat{\underline{f}}^T(\hat{\underline{\eta}}_1 t)}{\partial t}\right]^T \cdot \beta \, d\hat{\underline{\eta}} \qquad (2)$$

<u>Proof:</u> The transfer matrix $\hat{W}(s)$ (see Fig. 5) possesses a minimal realization $\{\hat{F},\hat{G},\hat{H}\}$ satisfying:

$$W(s) = \hat{H}^T \quad (sI-\hat{F})^{-1} \hat{G} \ ,$$

where

$$\hat{F} \equiv F \triangleq \left[\begin{array}{cc|cc} A_1 & 0 & & \\ 0 & A_1 & & \mathcal{O} \\ \hline & & A_2 & 0 \\ \mathcal{O} & & 0 & A_2 \end{array}\right] ; \quad \hat{G} = GT \triangleq \left[\begin{array}{cc} b_1 & 0 \\ 0 & b_1 \\ \hline 0 & b_2 \\ b_2 & 0 \end{array}\right] \cdot T$$

$$\hat{H}^T = T^{-1} H^T \triangleq T^{-1} \cdot \left[\begin{array}{cc|cc} c_1^T & 0 & c_2^T & 0 \\ 0 & c_1^T & 0 & c_2^T \end{array}\right] .$$

If:

$$\underline{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_4(t) \\ x_3(t) \\ x_2(t) \end{bmatrix} \qquad \text{and} \qquad \underline{f}[\underline{\sigma}(t),t] \triangleq \begin{bmatrix} f_1[\sigma_1(t),t] \\ f_2[\sigma_2(t),t] \end{bmatrix} ,$$

note: $\hat{\underline{f}}[\hat{\underline{\sigma}}(t),t] \triangleq T^{-1} \underline{f}[\underline{\sigma}(t),t]$

$\hat{\underline{\sigma}}(t) \triangleq T^{-1} \underline{\sigma}(t)$.

Then the state space representation of the system $\hat{S}$ that is characterized in Fig. 5 is:

$$\hat{S} \left| \begin{array}{l} \overset{o}{\underline{x}}(t) = F \quad \underline{x}(t) - \hat{G}\, \hat{\underline{f}}[\hat{\underline{\sigma}}(t),t] \\ \hat{\underline{\sigma}}(t) = \hat{H}^T \quad \underline{x}(t) \end{array} \right.$$

In order to prove lemma II, that supplies the sufficient condition for the stability of the system $\hat{S}$ (and S), we choose as a tentative Liapunov function:

$$V[x(t),t] = x^T(t) \quad x(t) + \int_0^{\sigma} \hat{\underline{f}}^T[\hat{\underline{\eta}},t] \quad \beta \; d\hat{\underline{\eta}} \qquad (3)$$

where

$$p(t) = p^T(t) > 0$$

But, from the definitions of $\hat{\underline{f}}$, $\hat{\underline{\sigma}}$ and T ,

$$V[x(t),t] = \underline{x}^T p \; \underline{x} + \int_0^{\sigma} [T^{-1} \quad \underline{f}(\underline{\eta},t)]^T \quad \hat{\beta} \quad d[T^{-1}\underline{\eta}] =$$

$$= \underline{x}^T p \; \underline{x} + \int_0^{\sigma} \underline{f}^T[\underline{\eta},t] \quad \{T\hat{\beta}T^{-1}\} d\underline{\eta}$$

$$= \underline{x}^T p \; \underline{x} + \int^{\sigma} \underline{f}^T(\underline{\eta},t) \quad \beta \; d\underline{\eta}$$

From the properties of the diagonal matrix β and that of P(t), it is obvious that the above choice of the tentative Liapunov function fulfills the conditions:

(1) $\underline{V}[\underline{x}.t] > 0, \ \forall \ \|\underline{x}\| \neq 0$

(2) $\underline{V}[o,t] = 0$

(3) $\underline{V}[\|\underline{x}\| \rightarrow \infty, t] = \infty$

Differentiating V[x(t),t] of equation 4 with respect to time, yields:

$$\frac{d}{dt}\underline{V}[\underline{x}(t),t] = \underline{\dot{x}}^T p\underline{x} + \underline{x}^T \dot{p}\underline{x} + \underline{x}^T p\underline{\dot{x}} + \hat{\underline{f}}^T(\hat{\underline{\sigma}},t)\ \beta\dot{\hat{\underline{\sigma}}} + \int_0^{\sigma}\left[\frac{\partial \underline{f}(\underline{\eta},t)}{\partial t}\right]^T \hat{\beta}\ d\hat{\underline{\eta}}\ .$$

Substituting for $\dot{x}(t)$ and collecting terms, yields:

$$\frac{d}{dt}\underline{V}[\underline{x}(t),t] = \underline{x}^T[Fp+pF+\dot{p}]x\ -2\underline{x}^T[p\hat{G} - 1/2F^T\hat{H}\hat{\beta}]-\hat{f}^T\hat{\beta}\hat{H}^T\hat{G}\hat{\underline{f}}$$

$$+\int_0^{\sigma}\left[\frac{\partial \hat{\underline{f}}(\hat{\underline{\eta}},t}{\partial t}\right]^T \hat{\beta}\ d\hat{\underline{\eta}}\ .$$

the term:

$$\hat{\underline{f}}^T(\hat{\underline{\sigma}},t)\quad \hat{\alpha}\quad \{\hat{\underline{\sigma}} - \hat{k}\quad \hat{\underline{f}}[\hat{\underline{\sigma}},t]\} \equiv$$

can be easily shown to equal the condition:

$$= \underline{f}^T[\underline{\sigma},t]\ \alpha\ \ \{\underline{\sigma}\text{-}k\ \underline{f}(\sigma,t)\} = \sum_i \alpha_i f_i[\sigma_i,t]\ \left[\sigma_i - \frac{f_i[\sigma_i,t]}{k_i}\right] \geq 0$$

and is added and subtracted, in the form:

$$2\underline{x}^T \left[\frac{1}{2} \hat{H} \hat{\alpha}\right] \hat{\underline{f}}[\hat{\underline{\sigma}},t] - \hat{\underline{f}}^T[\hat{\underline{\sigma}},t] \left\{\frac{1}{2} \hat{\alpha} \hat{K} + \frac{1}{2} \hat{K} \hat{\alpha}\right\} \hat{\underline{f}}[\hat{\underline{\sigma}},t]$$

$$- \hat{\underline{f}}^T[\hat{\underline{\sigma}},t] \alpha \left\{\hat{\alpha}\text{-}\hat{K} \hat{\underline{f}}(\hat{\underline{\sigma}},t)\right\} \equiv 0$$

from both sides of $\dot{V}$ to yield:

$$\dot{V}(x(t),t) = \underline{x}^T[Fp+pF+\dot{p}]\underline{x}-2\underline{x}^T\left[p\hat{G} - \frac{1}{2}(\hat{H} \hat{\alpha} + F^T\hat{H}\hat{\beta})\right] \hat{\underline{f}}(\hat{\underline{\sigma}},t)$$

$$- \hat{\underline{f}}^T(\hat{\underline{\sigma}},t)\left[\frac{1}{2}(\hat{\alpha} \hat{K} + \hat{\beta}\hat{H}^T\hat{G}) + \frac{1}{2}(\hat{\alpha} \hat{K} + \hat{\beta}\hat{H}^T\hat{G})^T\right] \hat{\underline{f}}(\hat{\underline{\sigma}},t) + \int_0^{\sigma} \left[\frac{\partial\hat{\underline{f}}(\hat{\underline{\eta}},t)}{\partial t}\right]^T \hat{\beta}\, d\hat{\underline{\eta}}$$

$$- \hat{\underline{f}}^T(\hat{\underline{\sigma}},t) \hat{\alpha} [\hat{\underline{\sigma}},t) \hat{\alpha} [\hat{\underline{\sigma}} - \hat{K} \hat{\underline{f}}(\hat{\underline{\sigma}},t)]$$

where $\frac{1}{2} [\hat{H} \hat{\alpha} + F^T\hat{H}\hat{\beta}] \hat{=} M$ and $\frac{1}{2} [\hat{\alpha} \hat{K} + \hat{\beta}\hat{H}^T\hat{G}] \triangleq J$

If we use, for the matrix function P(t) the solution of the differential equation:

$$- \dot{\pi}(t,t_1) = \pi[F-\hat{G}(J+J^T)^{-1} M^T] + [F^T-M(J+J^T)^{-1} \hat{G}^T] \pi$$

$$+ \pi \hat{G}(J+J^T)^{-1}\hat{G}^T\pi + M(J+J^T)^{-1}M^T , \quad \pi(t_1,t_1) = 0$$

which is guaranteed by lemma I, in the form $p(t) = \lim_{t_1 \to \infty} \pi(t,t_1)$, then it is easy to verify that:

$$\dot{p} + pF + F^T p = p\hat{G}(J+J^T)^{-1}M^T + M(J+J^T)^{-1}\hat{G}^T p$$

$$- p\hat{G}(J+J^T)^{-1}\hat{G}^T p - M(J+J^T)^{-1}M^T$$

$$\equiv = - (M-p\hat{G})(J+J^T)^{-1}(M^T-\hat{G}^T p) .$$

So that:

$$\hat{\dot{V}}[\underline{x}(t),t] = - \underline{x}^T(M-p\hat{G})(J+J^T)^{-1}(M^T-\hat{G}^T p)\ \underline{x}-Z\underline{x}^T(p\hat{G}-M)\hat{\underline{f}}(\hat{\underline{\sigma}},t)$$

$$- \hat{\underline{f}}^T(\hat{\underline{\sigma}},t)[J+J^T]\ \ \hat{\underline{f}}(\hat{\underline{\sigma}},t) - \hat{\underline{f}}(\hat{\sigma},t)\ \hat{\alpha}\ [\hat{\underline{\sigma}}-\hat{K}\ \underline{f}(\hat{\underline{\sigma}},t)] + \int_0^{\sigma}\left[\frac{\partial\hat{\underline{f}}(\underline{\eta},t)}{\partial t}\right]\hat{\beta}\ d\hat{\underline{\eta}}$$

and finally:

$$\dot{\underline{V}}[\underline{x}(t),t] = - \{\underline{x}^T[M-p\hat{G}](J+J^T)^{-1} - \hat{\underline{f}}(\hat{\underline{\sigma}}^T,t)\}(J+J^T)\ \{(J+J^T)^{-1}[M^T-G^T p]\underline{x}-\hat{\underline{f}}$$

$$- \left\{\hat{\underline{f}}^T(\underline{\sigma},t)\ \hat{\alpha}\ [\hat{\underline{\sigma}}-\hat{K}\ \hat{\underline{f}}(\hat{\underline{\sigma}},t)] - \int_0^{\sigma}\left[\frac{\partial\hat{\underline{f}}(\hat{\underline{\eta}},t)}{\partial t}\right]^T\hat{\beta}\ d\hat{\underline{\eta}}\right\} \qquad (4)$$

Clearly the first term (Eq. 4) is nonpositive, and if the conditions on the nonlinearity $\hat{\underline{f}}[\hat{\underline{\sigma}},t]$ are such that

$$\hat{\underline{f}}^T[\hat{\underline{\sigma}},t] \quad \hat{\alpha} \quad \{\hat{\underline{\sigma}}-\hat{K} \ \hat{\underline{f}}(\hat{\underline{\sigma}},t)\} > \int_0 \frac{\partial \hat{\underline{f}}^T(\hat{\underline{\eta}},t)}{\partial t} \ \hat{\beta} \ d\underline{\eta} \tag{5}$$

holds, then the nonpositive nature of $\dot{V}$ is guaranteed.

However, from the definitions of $\hat{\underline{f}}$, $\hat{\underline{\sigma}}$, $\hat{\alpha}$, $\hat{\beta}$, and $\hat{K}$ it is clear that the inequality [5] holds if:

$$\underline{f}^T[\sigma,t] \quad \alpha \quad [\underline{\sigma} - K \ \underline{f}(\sigma,t)] > \int \frac{\partial \underline{f}^T(\eta,t)}{\partial t} \ \beta \ d\underline{\eta}$$

where here α and β and K are diagonal matrices.

Theorem:

A) The symmetrical continuous-time, two-dimensional, nonlinear and time-varying autonomous systems of Fig. 1 is stable for:

$$\underline{f}^T[\underline{\sigma}(t),t] \quad \alpha[\underline{\sigma}(t) - K \ \underline{f}[\underline{\sigma}(t),t]] > \int_0^{\sigma} \frac{\partial \underline{f}^T(\underline{\eta},t)}{\partial t} \ \beta \ d\underline{\eta}$$

where $\quad \alpha = \text{diag}\{\alpha_1\alpha_2\}$, $\hat{\beta} = \text{diag}\{\beta_1\beta_2\}$, $K = \text{diag}\left\{\frac{1}{K_1} \ \frac{1}{K_2}\right\}$

if there exist scalars q_1 and q_2 such that:

$$(1)\quad \mathrm{Re}\{1+j\omega q_1\}\{H_1(j\omega) + H_2(j\omega)\} + \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \geq 0$$

$$(2)\quad \left[\mathrm{Re}\{1+j\omega q_1\}\{H_1(j\omega) + H_2(j\omega)\} + \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right] \cdot$$

$$\left[\mathrm{Re}\{1+j\omega q_2\}\{H_1(j\omega) - H_2(j\omega) \left\{+ \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right]\right.\right.$$

$$- \frac{1}{\alpha_1\alpha_2}\left[\frac{(\alpha_1 + \alpha_2)}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right]^2 \geq 0$$

$$(3)\quad q_j \int_0^{\sigma_j} \frac{\partial f_j[\eta_j(t),t]}{\partial t}\, d\eta_j < f_j[\sigma_j(t),t]\left[\sigma_j(t), \frac{f_j[\sigma_j(t),t]}{k_j}\right], j=1,2$$

(?) In the case where $k_1 = k_2 = k$ conditions 2, 3, and 4 are reduced to:

$$(1^1)\quad \mathrm{Re}\{1+j\omega q_1\}\{H_1(j\omega) + H_2(j\omega)\} + \frac{1}{k} \geq 0$$

$$(2^1)\quad \mathrm{Re}\{1+j\omega q_2\}\{H_1(j\omega) - H_2(j\omega)\} + \frac{1}{k} \geq 0$$

$$(3^1)\quad f_j[\sigma_j(t),t]\left[\sigma_j(t) - \frac{f_j(\sigma_j(t),t)}{K}\right] > q_j \int_0^{\sigma} \frac{\partial f_j(\eta_j,t)}{\partial t}\, d\eta_i$$

Proof: From applying Lemma II for the system $\hat{S}$, it is clear that the system $\hat{S}$ is stable for all $\underline{\hat{f}}[\underline{\hat{\sigma}}(t),t]$ satisfying the inequality:

$$\underline{\hat{f}}[\underline{\hat{\sigma}}(t),t] \; \hat{\alpha} \; \left\{\underline{\hat{\sigma}} - \hat{K} \; \underline{\hat{f}}[\underline{\hat{\sigma}}(t),t]\right\} > \int_0^{\sigma} \frac{\partial \underline{\hat{f}}[\hat{\underline{\eta}}^T(t),t]}{\partial t} \; \hat{\beta} \; d\hat{\underline{\eta}}$$

if there exist symmetrical positive semi-definite matrices $\hat{\alpha} \geq 0$, $\hat{\beta} \geq 0$, $(\hat{\alpha}+\hat{\beta}) > 0$ such that:

$$\hat{Z}(j\omega) + \hat{Z}^T(-j\omega) \geq 0$$

where:

$$\hat{Z}(s) \triangleq \hat{\alpha} \hat{K} + (\hat{\alpha} + \hat{\beta} s) W(s) \quad \text{is real}$$

The interpretation of these conditions, for the two inputs-two outputs case, in terms of $\underline{f}[\underline{\sigma}(t),t]$, K, $H_1(s)$ and $H_2(s)$ in Fig. 1 are:

$$K = \begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{bmatrix}$$

$$\hat{K} = \frac{1}{2} \left[\begin{array}{c:c} \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} - \frac{1}{k_2} \\ \hdashline \frac{1}{k_1} - \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \end{array}\right]$$

is clearly symmetric and positive definite.

ith this choice of R

$$2(s) = \left[\begin{array}{c:c} \frac{\alpha_1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) + (\alpha_1 + \beta_1 s)G_1(s) & \frac{\alpha_1}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) \\ \hdashline \frac{\alpha_2}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) & \frac{\alpha_2}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) + (\alpha_2 + \beta_2 s)G_2(s) \end{array}\right]$$

where

$$G_1(s) \triangleq H_1(s) + H_2(s)$$

$$G_2(s) \triangleq H_1(s) - H_2(s)$$

so that:

$$Z(j\omega)+Z(-j\omega)=\left[\begin{array}{c|c} a_1\left(\frac{1}{k_1}+\frac{1}{k_2}\right)+2\mathrm{Re}(\alpha_1+j\omega\beta_1)G_1(j\omega) & \frac{(\alpha_1+\alpha_2)}{2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right) \\ \hline \frac{\alpha_1+\alpha_2)}{2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right) & \alpha_2\left(\frac{1}{k_1}+\frac{1}{k_2}\right)+2\mathrm{Re}(\alpha_2+j\omega\beta_2)G_2(j\omega) \end{array}\right]$$

Now

$$Z(j\omega) + Z(-j\omega) \geq 0$$

Implies

1. $$\mathrm{Re}(1 + j\omega q_1)\{H_1(j\omega) + H_2(j\omega)\} + \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \geq 0$$

2. $$\left[\frac{(1/k_1 + 1/k_2)}{2} + \mathrm{Re}(1 + j\omega q_1)\{H_1(j\omega) + H_2(j\omega)\}\right]$$
$$\cdot\left[\frac{(1/k_1 + 1/k_2)}{2} + \mathrm{Re}(1 + j\omega q_1)\{H_1(j\omega) + H_2(j\omega)\}\right]$$
$$- \frac{1}{\alpha_1\alpha_2}\left[\frac{\alpha_1+\alpha_2)}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right]^2 \geq \alpha$$

Condition 3) is directly derived from applying the orthogonal transformation T (see section II), so that

$$\hat{\underline{f}}^T[\sigma,t] \;\; \alpha \;\; \{\hat{\underline{\sigma}} - \hat{K}\, \underline{f}[\hat{\sigma},t]\} = \left[T^{-1} \;\; \underline{f}[\underline{\sigma},t]\right]^T \;\; \hat{\alpha} \;\; \left\{T^{-1}\;\underline{\sigma}-\hat{K}T^{-1}\underline{f}[\sigma,t]\right\}$$

$$= \underline{f}^T[\underline{\sigma},t] \;\; \alpha \;\; \{\underline{\sigma}-K\underline{f}[\underline{\sigma},t]\}$$

$$= \sum_{j=1}^{2} \alpha_j \; f_j[\sigma_j(t),t] \; \left\{\sigma_j(t) - \frac{f_j[\sigma_j(t),t]}{K_j}\right\}$$

and in the same way:

$$\int_0^{\hat{\sigma}} \left[\frac{\hat{\underline{f}}[\hat{\underline{\eta}}(t),t]}{\partial t} \right]^T \hat{\beta} \quad d\hat{\underline{\eta}} = \int_0^{\sigma} \left[\frac{f[\underline{\eta}(t),t]}{\partial t}\right]^T \quad \beta \; d\underline{\eta} = \sum_{j=1}^{2} \beta_j \int^{\sigma_j} \frac{\partial f_j[\eta_j(H,t]}{\partial t} \quad d\eta_j$$

to yield:

$$(3) \quad g_j \int_0^{\sigma_j} \frac{\partial f_j[\eta_j(t),t]}{\partial t} \; d\eta_j < f_j[\sigma_j(t),t] \; \left\{\sigma_j(t) - \frac{f_j[\sigma_j(t),t]}{k_j}\right\}$$

where $g_j \triangleq \frac{\beta_j}{\alpha_j}$, $(j = 1,2)$.

Conditions (1^1), (2^1) and (3^1) follow by substituting $K_1 = K_2 \triangleq K$.
Thus the theorem is proved.

IV. Geometric Interpretation of Results for $k_1 = k_2 \triangleq k$

The conditions 1),2) can now be interpreted in the Popov plane. The forbidden region which the frequency responses $G_1(j\omega) \triangleq H_1(j\omega) + H_2(j\omega)$ and $G_2(j\omega) = H_1(j\omega) - H_2(j\omega)$ are not allowed to enter is shown in Fig. 7. The intersection of the Popov lines with the negative real axis is common to both systems and gives the numerical value of k, provided that condition 3^2 holds. Condition 3^1 gives the trade-off between the nonlinearities and their time derivative under which the stability (Popov type) still holds. This result is a generalization of the one reported in [5].

From the analysis it is evident that the stability properties of the given coupled system depend on the characteristics of the coupling $H_2(s)$. The stability of the system can be improved by the introduction of such coupling, $H_2(s)$, that shifts simultaneously the modified polar plots of $H_1(j\omega) + H_2(j\omega)$ and $H_1(j\omega) - H_2(j\omega)$ to the right of the previous Popov lines, such that the new intersection of the Popov lines with the negative real axis is closer to the origin.

V. Conclusion

In this paper an improved stability criterion has been derived and for the symmetric two-dimensional nonlinear time varying system. This criterion allows[1] autonomous, continuous-time. The intersections of the lines with the real axis are the same and give the maximum allowable gain of the nonlinearities. The flexibility in choosing the slopes results in an improved gain factor.[2] Some consideration is also given to the effect of the cross coupling on the stability of the one-dimensional system (without cross-coupling). It can be concluded that the stability of a one-dimensional system $H_1(s)$ can be improved by introducing a cross coupling $H_2(s)$ if the Popov plots of $H_1(s) + H_2(s)$ and $H_1(s) - H_2(s)$ are to the right of the plot of $H_1(s)$ in the Popov plane, as long as these slopes are positive.

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[1] Extention of the Popov Criterion to nonlinearities that are time varying as well, provided that certain trade-off between the non-linear gain and its time derivative is maintained.

[2] From one side but also puts some restrictions on the trade-off between the nonlinear element and its time derivative, so that the slopes of the Popov lines have an additional physical meaning that supplies the actual above mentioned, trade-off.

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